DIFFERENTIAL NOTE

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Computing antiderivatives is the simplest case of solving differential equations. If you have to do this frequently, it is convenient to have formulas for the antiderivative of common functions. In this note we consider the problem of how to find the antiderivative of $f(x)e^x$ for a general function $f(x)$. Let's try some examples.

Example 1. Suppose $f(x) = x$. What's the antiderivative of xe^x? This is one of those functions where your gut tells you to integrate by parts.

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C.$$

So the antiderivative of xe^x is $(x-1)e^x + C$.

Example 2. Next let's try $f(x) = x^2$. We compute the antiderivative of x^2e^x using integration by parts *twice* (although only once, really, if we use the answer we found in the first example.)

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 - 2(x-1)e^x + C = (x^2 - 2x + 2)e^x + C.$$

Example 3. Extrapolating from the previous two examples, we could find the antiderivative of $x^n e^x$ for any particular n integrating by parts n times, although that seems painful. If only there were another way...

Example 4. What about a function like $f(x) = \sin(x)$? We should still integrate by parts.

$$\int \sin(x)e^x dx = \sin(x)e^x - \int \cos(x)e^x dx.$$

This doesn't look much better, but we can also try integrating the right hand side by parts.

$$\int \cos(x)e^x dx = \cos(x)e^x + \int \sin(x)e^x dx.$$

Moving all the integrals in the last two equations to the left hand side we have a system of equations with two unknowns:

$$\int \cos(x)e^x dx + \int \sin(x)e^x dx = \sin(x)e^x$$
$$\int \cos(x)e^x dx - \int \sin(x)e^x dx = \cos(x)e^x.$$

Solving for the integrals (and tossing in the $+C$) we find

$$\int \cos(x)e^x dx = \tfrac{1}{2}\big(\sin(x) + \cos(x)\big)e^x + C$$
$$\int \sin(x)e^x dx = \tfrac{1}{2}\big(\sin(x) - \cos(x)\big)e^x + C.$$

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Example 5. What if $f(x) = x^{-1} = \frac{1}{x}$? Following the lead of the previous examples we try integration by parts:

$$\int x^{-1}e^x dx = x^{-1}e^x + \int x^{-2}e^x dx.$$

Oh no! It looks like the integrand just got worse. What do we do? We will return to this example.

Integrating by parts all the time is a hassle. It would be nice to have another way to compute

$$\int f(x)e^x dx$$

for a general function $f(x)$. In the next section we do just that.

Working with D. We write D for the *differential operator* $\frac{d}{dx}$. So if $f(x)$ is a differentiable function, then $Df(x) = f'(x)$. Powers of D correspond to taking multiple derivatives. That is, $D^2 f(x) = f''(x)$ and more generally $D^n f(x) = f^{(n)}(x)$. The benefit of this notation is that we feel better algebraically manipulating D, which as we will see, is the key to solving this whole problem.

Notice that if $f(x)$ is any differentiable function, the product rule tells us

$$D\big(f(x)e^x\big) = f(x)e^x + f'(x)e^x = \big(f(x) + f'(x)\big)e^x.$$

Rewriting this last expression with the D notation we have

$$\big(f(x) + f'(x)\big)e^x = \big(f(x) + Df(x)\big)e^x = \big((1 + D)f(x)\big)e^x.$$

Putting it all together gives the simple identity

$$D\big(f(x)e^x\big) = \big((1 + D)f(x)\big)e^x. \tag{1}$$

Check out what happens we take differentiating both sides of (1):

$$D^2\big(f(x)e^x\big) = D\big((1 + D)f(x)\big)e^x = \big((1 + D)^2 f(x)\big)e^x.$$

What is this really saying? It's a very compact formula: to use it first expand $(1 + D)^2$ to get $1 + 2D + D^2$; then

$$D^2\big(f(x)e^x\big) = \big((1 + 2D + D^2)f(x)\big)e^x = \big(f(x) + 2f'(x) + f''(x)\big)e^x.$$

Taking more derivatives, the pattern continues to hold:

$$D^n\big(f(x)e^x\big) = \big((1 + D)^n f(x)\big)e^x.$$

To appreciate how clean this formula is, note that if we expand $(1+D)^n$ using the binomial theorem we get the more intimidating identity

$$D^n\big(f(x)e^x\big) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)e^x.$$

Anyways, back to our original problem. Knowing that the derivative of a function in the form $f(x)e^x$ also has this form gives us a hint that perhaps an antiderivative of $f(x)e^x$ may also have this form. This is the logic behind the so-called *method of undetermined coefficients*. So suppose there were a function $g(x)$ such that

$$D\big(g(x)e^x\big) = f(x)e^x.$$

Then (1) tells us

$$f(x)e^x = D\big(g(x)e^x\big) = \big((1 + D)g(x)\big)e^x.$$

Dividing by e^x we get the simpler equation

$$f(x) = (1 + D)g(x).$$

If we want to solve for $g(x)$, our algebraic impulse is to divide both sides of this equation by $(1+D)$. We should be a little careful. But let's throw caution to the wind and just divide by $(1+D)$ anyways:

$$g(x) = \frac{1}{1+D} f(x). \tag{2}$$

How can we make sense of $\frac{1}{1+D}$? Recall the geometric series identity:

$$\frac{1}{1+D} = 1 - D + D^2 - D^3 + D^4 - \ldots = \sum_{k\geq 0} (-1)^k D^k.$$

What this means algebraically is that multiplying by $\frac{1}{1+D}$ is *the same* as multiplying by the power series $1 - D + D^2 - D^3 + \ldots$. We know how to multiply powers of D times a function $f(x)$, that just means taking a bunch of derivatives. So we rewrite (2) as

$$g(x) = \sum_{k\geq 0} (-1)^k D^k f(x),$$

which is valid whenever the right hand side of the equation makes sense (i.e. converges.) To summarize, we derived the following:

Theorem 6. *If $f(x)$ is an infinitely differentiable function, then*

$$\int f(x)e^x dx = \Big(\frac{1}{1+D} f(x)\Big)e^x + C = \Big(\sum_{k\geq 0}(-1)^k D^k f(x)\Big)e^x + C.$$

Let's put Theorem 6 to work.

Example 7. Suppose $f(x) = x$. Then $D^k x = 0$ for $k \geq 2$, so

$$\sum_{k\geq 0} (-1)^k D^k x = (1 - D + D^2 - D^3 + \ldots)x = (1-D)x = x - 1,$$

hence $\int xe^x dx = (x-1)e^x + C$, which agrees with our original computation.

Example 8. If $n \geq 0$ is an integer, then $D^k x^n = 0$ when $k > n$. Therefore

$$\int x^n e^x dx = \Big(\sum_{k\geq 0}(-1)^k D^k x^n\Big)e^x + C = \Big(\sum_{k=0}^{n}(-1)^k D^k x^n\Big)e^x + C.$$

For example, say $f(x) = x^5$, then

$$\begin{aligned}\int x^5 e^x dx &= \big((1 - D + D^2 - D^3 + D^4 - D^5)x^5\big)e^x + C \\ &= (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C.\end{aligned}$$

That was much easier than integrating by parts five times.

Example 9. We need a little finesse to apply Theorem 6 to $\int \sin(x)e^x dx$ or $\int \cos(x)e^x dx$ because the derivatives of $\sin(x)$ and $\cos(x)$ repeat periodically. Notice that

$$D^2 \sin(x) = -\sin(x) \Longrightarrow (1 - D^2)\sin(x) = 2\sin(x).$$

Hence

$$\tfrac{1}{1+D}\sin(x) = \tfrac{1}{2}\tfrac{1}{1+D} 2\sin(x) = \tfrac{1}{2}\tfrac{1-D^2}{1+D}\sin(x) = \tfrac{1}{2}(1-D)\sin(x) = \tfrac{1}{2}\big(\sin(x) - \cos(x)\big)$$

So we find

$$\int \sin(x)e^x dx = \big(\tfrac{1}{1+D}\sin(x)\big)e^x + C = \tfrac{1}{2}\big(\sin(x) - \cos(x)\big)e^x + C.$$

A similar trick works for $\int \cos(x)e^x dx$, can you see how to do it?

Example 10. Let's apply Theorem 6 to $\int x^{-1}e^x dx$. Computing the first few terms of $\frac{1}{1+D}x^{-1}$, a pattern becomes evident:

$$\tfrac{1}{1+D}x^{-1} = \tfrac{1}{x} + \tfrac{1}{x^2} + \tfrac{2}{x^3} + \tfrac{6}{x^4} + \ldots = x^{-1}\sum_{k\geq 0}\frac{k!}{x^k}.$$

This infinite series looks like an upside down version of the Taylor expansion of e^x,

$$e^x = \sum_{k\geq 0}\frac{x^k}{k!}.$$

You might ask yourself "what function is that?" I don't know that it has a name, as it isn't much good as a function on $\mathbb{R}$: this power series diverges everywhere. However, I can think of a couple reasons this series might be of interest to a number theorist. What does this tell us about $\int x^{-1}e^x dx$? It means there is no function $g(x)$ so that $D\big(g(x)e^x\big) = x^{-1}e^x$. But what if you really need to integrate $x^{-1}e^x$? In that case you can always expand $x^{-1}e^x$ as a Taylor series and integrate one term at a time.

$$\int x^{-1}e^x dx = \int \big(x^{-1} + 1 + \frac{x}{2} + \frac{x^2}{6} + \ldots\big)dx = \log(x) + x + \frac{x^2}{4} + \frac{x^3}{18} + \ldots.$$

What is the series

$$x + \frac{x^2}{4} + \frac{x^3}{18} + \ldots = \sum_{k\geq 1}\frac{x^k}{k\cdot k!}?$$

It isn't a function with a common name. When this happens you can simply *invent* a name. That's how all the functions we know got their names. The psychological effect of this slight of hand should not be underestimated. Say we call it the *alpha function*

$$\alpha(x) = \sum_{k\geq 1}\frac{x^k}{k\cdot k!}.$$

Then when someone asks you to compute $\int x^{-1}e^x dx$ you can say

$$\int x^{-1}e^x dx = \log(x) + \alpha(x) + C.$$

The problem went from "having no solution" to having an easy one.

Example 11. For our final example let $f(x) = e^{rx}$. We can easily do this by hand:

$$\int f(x)e^x dx = \int e^{rx}e^x dx = \int e^{(r+1)x}dx = \tfrac{1}{r+1}e^{(r+1)x} + C.$$

Nevertheless, Theorem 6 *still works*! From

$$\begin{aligned}\tfrac{1}{1+D}e^{rx} &= (1 - D + D^2 - D^3 + \ldots)e^{rx}\\ &= e^{rx} - re^{rx} + r^2e^{rx} - r^3e^{rx} + \ldots\\ &= (1 - r + r^2 - r^3 + \ldots)e^{rx}\\ &= \tfrac{1}{r+1}e^{rx},\end{aligned}$$

we compute

$$\int f(x)e^x dx = \big(\tfrac{1}{1+D}e^{rx}\big)e^x + C = \tfrac{1}{r+1}e^{rx}e^x + C = \tfrac{1}{r+1}e^{(r+1)x} + C.$$

Math is, once again, consistent.

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